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ADAPTIVE TRACKING FOR SCALAR MINIMUM PHASE SYSTEMS

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Abstract
We present the concept of a universal adaptive tracking controller for classes of linear systems. For the class of scalar minimum phase systems of relative degree one, adaptive tracking is shown for arbitrary finite dimensional reference signals. The controller requires no identification of the system parameters. Robustness properties are explored.

1. INTRODUCTION

In general terms, the basic problem of parameter adaptive control may be described as follows:

"Suppose a black box is given whose internal dynamics is only roughly understood; for example one might know only certain bounds on the (possibly time varying) system parameters but not the precise parameter values. Furthermore, some structural properties are supposed to be known, so that one can specify a model class the unknown system belongs to. One wants to control the system by an explicit controller which is capable of learning enough through the observed output y(t) and input u(t), resp., of the system, to achieve its control purpose." (cf. for example [8] for a precise formalization of this heuristic definition.)

Such a control objective might be for example that of model reference adaptive control (MRAC), where the output $y(\cdot)$ to be controlled is required to track the output $r(\cdot)$ of a prescribed linear model for a certain class of admissible control functions and initial conditions. The traditional approach to this problem is based on system identification where parameter optimization techniques are used to obtain approximate values of the (unknown) system parameters, together with a conventional controller design scheme; see e.g. [5]. However, these "mixed" MRAC-algorithms work only under rather strong a priori assumptions on the systems to be controlled and due to the intrinsic computational complexity of the used identification scheme, these adaptive controllers are complicated both from a theoretical as well as computational point of view.

Moreover, for biological systems it is sometimes more important to explain a certain dynamical behavior rather than to construct a dynamical process as in technological systems. For this purpose it is in general meaningless to include an "identification box" in an adaptive feedback loop. Instead, the feedback mechanisms are often just some nonlinear functions which change the open loop dynamics such that the observed behavior can be explained.

It is therefore of considerable practical as well as theoretical importance to construct adaptive control schemes without explicit identification of the system parameters. Particular algorithms for such simplified controllers have been proposed in [1] and [2]. In a more systematic framework adaptive controllers which require no explicit identification and which stabilize linear systems under very weak assumptions on the system parameters have been developed in [4], [6], [7], [9], [10] and [11]. These adaptive controllers are called universal, since they achieve their control objective for a whole prescribed class of linear systems and all possible initial conditions.

Previous work on universal adaptive controllers was mainly concerned with the adaptive stabilization problem. The purpose of this paper is to extend these results to the adaptive tracking problem:

"Derive for a given class of reference signals $r(\cdot) \in R$ and a class Σ of linear systems an adaptive feedback controller such that for every system in Σ its output asymptotically tracks $r(\cdot)$."

To formalize this task let $\Sigma = \Sigma(m,p)$ be a class of linear time-invariant systems

$$\dot{x}(t) = Ax(t)+Bu(t) \ , \quad y(t) = Cx(t)$$
$$x(t) \in \mathbb{R}^n \ , \quad u(t) \in \mathbb{R}^m \ , \quad y(t) \in \mathbb{R}^p$$

with m, p given and n arbitrary.

Let

$$R \subset C_{pc}([0,\infty),\mathbb{R}^p)$$

be a prescribed class of possible (piecewise continuous) reference signals $r(\cdot)$ and let

$$\mathcal{M} \subset C_{pc}([0,\infty)\mathbb{R}^p)$$

be a prescribed <u>measurement space</u>.

A <u>universal adaptive tracking controller</u> (UATC) for (Σ,R,M) consists of

- a parameter space $\mathbb{R}^q$ for the feedback gains $k(t)$.
- a smooth control law

  $$u(t) = f(t,k(t),y(t),r(t))$$

  where $f\colon \mathbb{R}^{1+q+2p} \longrightarrow \mathbb{R}^m$ is C^∞.
- a parameter adaptation law

  $$k(\cdot) = \alpha(y(\cdot),r(\cdot)) \ ,$$

  where

  $$\alpha\colon C_{pc}([0,\infty),\mathbb{R}^p)^2 \longrightarrow C_{pc}([0,\infty),\mathbb{R}^q)$$

  is causal such that for any $(A,B,C) \in \Sigma$, any initial data $x(0)$, $k(0)$ and any $r(\cdot) \in R$ a unique solution of the closed loop system

  $$\text{(CL)} \qquad \begin{aligned} \dot{x}(t) &= Ax(t)+Bf(t,k(t),y(t),r(t)) \\ y(t) &= Cx(t) \\ k(\cdot) &= \alpha(y(\cdot),r(\cdot)) \end{aligned}$$

  exists for all $t \geq 0$ and satisfies

  - $k(\cdot) \in L_\infty([0,\infty),\mathbb{R}^q)$
  - $e(\cdot) := y(\cdot)-r(\cdot) \in M$.

In the sequel we ignore the question of uniqueness and existence of solutions of (CL), which would follow by a suitable regularity assumption on α.

In this paper we assume that the system class is the set $\Sigma(1,1)$ of all <u>scalar minimum phase systems</u> (A,b,c) <u>with relative degree</u>

one:

- $m = p = 1$
- $cb \neq 0$
- $\det\begin{bmatrix} sI-A & b \\ c & 0 \end{bmatrix}$ a Hurwitz polynomial

2. ROBUST ADAPTIVE TRACKING WITHOUT SWITCHING

In this section we consider the subclass $\Sigma_+ \subset \Sigma(1,1)$ of systems (A,b,c) with

$$cb > 0 \tag{2.1}$$

The following high gain theorem is well known; see [6], [10], [3]:

2.1 High Gain Theorem

Let $k(\cdot) \in C_{pc}([0,\infty),\mathbb{R})$ be monotonically increasing with $k_\infty = \lim_{t\to\infty} k(t) = +\infty$ and $(A,b,c) \in \Sigma_+$. Then the closed loop system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{2.2a}$$

$$u(t) = -k(t)\, cx(t) \tag{2.2b}$$

is exponentially stable. □

Let

$$\|\cdot\|_T : C_{pc}([0,T],\mathbb{R}^p) \longrightarrow \mathbb{R}, \quad T \geq 0$$

be a family of functionals satisfying for every $f \in C_{pc}([0,\infty),\mathbb{R}^p)$ the truncation condition:

(A1) $\quad S > T \Rightarrow \|f\|_S \geq \|f\|_T$

where

$$\|f\|_S := \|f_{|[0,S]}\|_S \;.$$

The choice of the measurement space M is open to the designer to produce the desirable responses. The only constraints which we impose (cf. [8]) are:

$$E([0,\infty),\mathbb{R}^p) \subset M \subset C_{pc}([0,\infty),\mathbb{R}^p)$$

where
$$E([0,\infty),\mathbb{R}^p) := \Big\{ y(\cdot) \in C_{pc}([0,\infty),\mathbb{R}^p) \;\; \exists\, M,\alpha \in \mathbb{R}_+ \;;\;\; \|y(t)\| \leq Me^{-\alpha t} \Big\} \;.$$

Furthermore M has to satisfy the "completeness" condition:

(Δ2) $\qquad \lim\limits_{T\to\infty} \|f\|_T < \infty \quad$ if and only if $\quad f \in M$

If we now provide the closed loop system (2.2) with the gain adaptation law:

$$k(t) = \|y(\cdot)\|_t \;, \quad t \in \mathbb{R}_+ \tag{2.3}$$

then we obtain as an immediate consequence of the HGT the following

2.2 High Gain Adaptive Stabilization Theorem

Given Σ_+, M and a family of functionals $\|\cdot\|_T$, $T \in \mathbb{R}_+$, satisfying (Δ), then for every $(A,b,c) \in \Sigma_+$ and initial data, the solutions $y(\cdot)$ of

(CLAS)
$$\begin{aligned} \dot{x}(t) &= (A-k(t)bc)x(t) \\ y(t) &= cx(t) \\ k(t) &= \|y(\cdot)\|_t \end{aligned}$$

satisfy:

$$y(\cdot) \in M \tag{2.4a}$$

$$\lim_{t\to\infty} k(t) = k_\infty < \infty \quad \text{exists} \tag{2.4b}$$

□

2.3 Remarks

a) In the above theorem it is implicitly presupposed that the family $\|\cdot\|_T$, $T \in \mathbb{R}_+$, is selected such that the solutions of (CLAS) exist. Examples for which this is the case are:

$$\|f\|_t = \gamma + \int_0^t |f(s)|^p ds \quad \text{for} \quad p<\infty \tag{2.5a}$$

$$\|f\|_t = \gamma + \max_{0\le s\le t} |f(s)| \tag{2.5b}$$

b) In [4,6,10] it is shown that in the case of L_p-adaptation laws with $p=2q$

$$k(t) = \int_0^t |y(s)|^{2q} ds + \gamma$$

the output $y(t)$ of (CLAS) goes asymptotically to zero

$$\lim_{t\to\infty} y(t) = 0 \ .$$

c) By a theorem of Lebesgue, every monotone function $k: \mathbb{R}_+ \to \mathbb{R}$ is differentiable outside a subset of $\mathbb{R}_+$ of measure zero. Thus almost everywhere

$$\dot{k}(t) = F(t,y(\cdot))$$

with

$$F(t,y(\cdot)) = \frac{d}{dt}\|y(\cdot)\|_t \ .$$

□

The stabilization property of the above adaptive controller is robust against additive input or state disturbances $d(t)$ which are related in the following way to the measurement spaces M:

"Let D_ℓ be a linear subspace of $C_{pc}([0,\infty),\mathbb{R}^\ell)$ such that for any exponentially stable operator $\Phi(t,s)$ and $d \in D_\ell$

$$\Phi * d \in M \tag{2.6}$$

where

$$(\Phi * d)(T) = \int_0^T \Phi(T,s)d(s)ds \ .$$

2.4 Robustness Theorem

Let $(M, \|\cdot\|_T)$ be a given measurement space satisfying (Δ) and let D_n be a class of disturbances satisfying (2.6). Then for all $(A,b,c) \in \Sigma_+$, $x(0) = x_o \in \mathbb{R}^n$, $d(\cdot) \in D_n$ the solutions of

$$\dot{x}(t) = (A-k(t)bc)x(t) + d(t)$$

$$y(t) = cx(t) \ , \quad k(t) = \|y(\cdot)\|_t$$

satisfy

$$y(\cdot) \in M \tag{2.4a}$$

$$\lim_{t\to\infty} k(t) = k_\infty < \infty \quad \text{exists} \ . \tag{2.4b}$$

__Proof:__ By $(\Delta 1)$, $k(t)$ is monotonically increasing. Suppose $k(t)\to\infty$ as $t\to\infty$. Then by the HGT the system $\dot{\xi}(t) = (A-k(t)bc)\xi(t)$ is exponentially stable. Because $E \subset M$ and by (2.6) $y = cx \in M$ and therefore by $(\Delta 2)$ $k(\cdot) \in L_\infty([0,\infty),\mathbb{R})$, in contradiction to $k(t) \to \infty$. Thus $\lim_{t\to\infty} k(t) < \infty$ exists and hence by $(\Delta 2)$ $y(\cdot) \in M$.

□

Consider now the class Σ_+ and let $R_{p(s)}$, $p(s) = s^\ell + p_{\ell-1}s^{\ell-1} + \ldots + p_o \in \mathbb{R}[s]$, be the solution space of the differential equation

$$p(\tfrac{d}{dt})r(\cdot) = 0 \ .$$

For the tracking problem it is now required that the tracking error $e(t) = y(t)-r(t)$ tends to 0 if $t\to\infty$:

$$\lim_{t\to\infty} e(t) = 0 \tag{2.7}$$

In the nonadaptive case the tracking problem can be viewed as a stabilization problem by inserting the dynamics of the reference signal into the dynamics of the given system. This is done by augmenting the system with a suitable <u>prefilter of relative degree zero</u>. We show that in the adaptive case this idea works too.

Choose a monic Hurwitz polynomial $q(s)$ of the same degree ℓ as $p(s)$. Find a state space realization (A_r, b_r, c_r, d_r), $d_r=1$, of the transfer function $\frac{q(s)}{p(s)}$. u shall be the output of this system, the input is denoted by v. We obtain the following state space equation for the augmented system:

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}v \tag{2.8a}$$

$$y = \hat{c}\hat{x} \ , \tag{2.8b}$$

where $\hat{A} = \begin{bmatrix} A & bc_r \\ 0 & A_r \end{bmatrix}$, $\hat{b} = \begin{bmatrix} b \\ b_r \end{bmatrix}$, $\hat{c} = [c \quad 0]$, $\hat{x} = \begin{bmatrix} x \\ x_r \end{bmatrix}$, and

$(A,b,c) \in \Sigma_+$ denotes the system under consideration. Since $p(\frac{d}{dt})r(\cdot) = 0$, also $\chi(\frac{d}{dt})r(\cdot) = 0$, where χ is the characteristic polynomial of $\hat{A}$. If we assume that $(\hat{A},\hat{c})$ is observable, then $\hat{c}e^{\hat{A}t}$ is a fundamental system for the differential equation $\chi(\frac{d}{dt})r(\cdot) = 0$, and $r(\cdot)$ can therefore be generated by $(\hat{A},\hat{c})$ through an appropriate initial state $\bar{x}(0)$:

$$\dot{\bar{x}}(t) = \hat{A}\bar{x}(t)$$

$$r(t) = \hat{c}\bar{x}(t) \ .$$

This yields $(\dot{\hat{x}-\bar{x}}) = \hat{A}(\hat{x}-\bar{x})+\hat{b}v$

$$e = \hat{c}(\hat{x}-\bar{x})$$

with $(\hat{A},\hat{b},\hat{c}) \in \Sigma_+$, too.

Now choose:

$$v(t) = -k(t)\; e(t) \tag{2.9a}$$

$$k(t) = \|e(\cdot)\|_t \tag{2.9b}$$

as in theorem 2.2. This amounts to the following choice of $\hat{f}$ and $\hat{\alpha}$ in the construction of an UATC for Σ_+:

- feedback gains $z = \begin{pmatrix} x_r \\ k \end{pmatrix} \in \mathbb{R}^{\ell+1}$

- control law

$$\begin{aligned} u(t) &= \hat{f}(t,k(t),y(t),r(t)) \\ &= c_r x_r(t) - k(t)(y(t)-r(t)) \\ &= [c_r,-e(t)]z(t) \end{aligned} \tag{2.10a}$$

- parameter adaptation law

$$z(\cdot) = \hat{\alpha}(y(\cdot),r(\cdot))$$

with

$$z(t) = \begin{pmatrix} x_r(t) \\ k(t) \end{pmatrix}$$

$$= \begin{bmatrix} e^{A_r t} x_r(0) - \int_0^t e^{A_r(t-s)} b_r \|e(\cdot)\|_s e(s)\,ds \\ \|e(\cdot)\|_t \end{bmatrix} \tag{2.10b}$$

The resulting closed loop system is

$$\dot{x}(t) = Ax(t)+b(c_r x_r(t) - \|e(\cdot)\|_t \cdot e(t)) \tag{2.11a}$$

$$\dot{x}_r(t) = A_r x_r(t)-b_r \|e(\cdot)\|_t \cdot e(t)\ . \tag{2.11b}$$

We have thus obtained the following adaptive tracking result as a corollary of Theorem 2.2.

2.5 Corollary

Given $\Sigma = \Sigma_+$, $R = R_{p(s)}$, with $p(s) \in \mathbb{R}[s]$ monic of degree ℓ and $(M, \|\ \|_T)$ satisfying (Δ). Then $(\hat{f},\hat{\alpha})$ given by (2.10) is a $(\ell+1)$-parameter UATC for $(\Sigma_+, R_{p(s)}, M)$.

□

2.6 Remarks

a) If p=2q is even in the adaptation law (2.5a) then the error e(t) asymptotically tends to 0:

$$\dot{k}(t) = |e(t)|^{2q} \Rightarrow \lim_{t\to\infty} e(t) = 0 \tag{2.12}$$

b) We have proved Corollary 2.5 only in the case where p(s) and n(s), the numerator of $g(s) = c(sI-A)^{-1}b = \frac{n(s)}{d(s)}$, are coprime. But because n(s) is Hurwitz, common factors of p(s) and n(s) are itself stable and the corresponding modes of (A_r, b_r, c_r, d_r) are exponentially stable, hence the tracking result (ii) remains true in this case.

c) Let $\mathcal{D}_n$ be a class of disturbances satisfying (2.6). Corollary (2.5) remains true if the original system $(A,b,c) \in \Sigma_+$ is perturbed by arbitrary disturbances $d(\cdot) \in \mathcal{D}_n$. In particular, with the gain function

$$k(t) = \|e(\cdot)\|_t := \gamma + \max_{0\le s\le t} \|e(s)\| ,$$

the closed loop system

$$\dot{x}(t) = Ax(t) + b(c_r x_r(t) - \|e(\cdot)\|_t e(t)) + d(t)$$

$$\dot{x}_r(t) = A_r x_r(t) - b_r \|e(\cdot)\|_t e(t)$$

satisfies

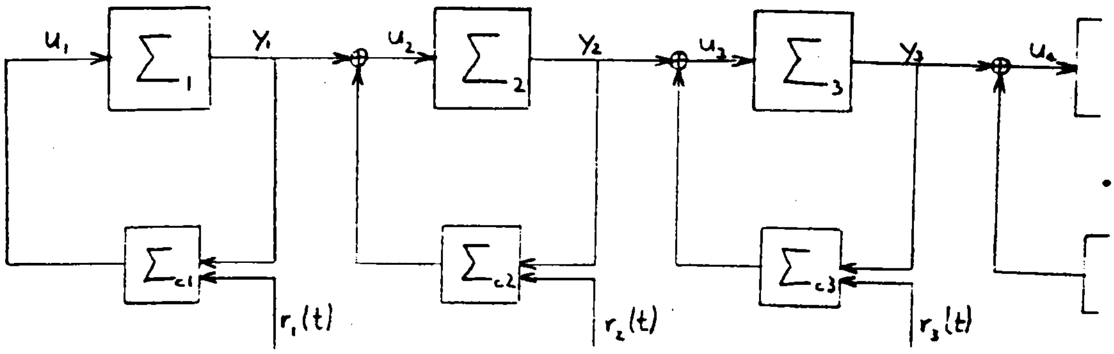
- $k(\cdot) \in L_\infty([0,\infty),\mathbb{R})$
- $e(\cdot) = y(\cdot) - r(\cdot) \in L_\infty([0,\infty),\mathbb{R})$.

Hence the output tracking error remains bounded for arbitrary bounded disturbances. Similarly for L_p-disturbances (consequence of Thm. (2.4)).

Finally as an application of the previous result we consider a <u>series coupling of systems</u> belonging to the class Σ_+. Let $r_i(\cdot)$

be the reference signal for the i-th system in the coupling with

$$\tilde{p}_i(\tfrac{d}{dt})r_i(\cdot) = 0, \quad \tilde{p}_i(s) \in \mathbb{R}[s] \quad \text{arbitrary} \qquad (2.13)$$
$$i = 1,\ldots,N$$



(2.14)

In order to construct an adaptive tracking controller for this series connection we define polynomials

$$p_1(s) := \tilde{p}_1(s) \qquad (2.15a)$$

$$p_{i+1}(s) := \ell cm(\tilde{p}_i(s),\tilde{p}_{i+1}(s)) \qquad (2.15b)$$
$$i = 1,\ldots,N-1$$

and select local controllers $(\hat{f}_i,\hat{\alpha}_i)$, $i=1,\ldots,N$ of the form (2.10) for the subsystems Σ_i. More precisely, for the first system (A_1,b_1,c_1) we choose $(\hat{f}_1,\hat{\alpha}_1)$ as in Corollary (2.5). The tracking error $e_1(\cdot)$ then satisfies the prescribed tracking requirement $e_1(\cdot) \in M$. Now for the system $(i+1)$, $(A_{i+1},b_{i+1},c_{i+1})$, there is an external "input" $y_i(\cdot)$, the output of system i. Assume, tracking was performed successfully for system i, $e_i(\cdot) = y_i(\cdot)-r_i(\cdot) \in M$. Let $(A^r_{i+1},b^r_{i+1},c^r_{i+1},d^r_{i+1} = 1)$ be constructed as for Corollary 2.4 with $p_{i+1}(s)$ defined by (2.15). Then we obtain

$$\begin{pmatrix} \dot{x}_{i+1} \\ \dot{x}^r_{i+1} \end{pmatrix} = \begin{pmatrix} A_{i+1} & b_{i+1} \cdot c^r_{i+1} \\ 0 & A^r_{i+1} \end{pmatrix} \begin{pmatrix} x_{i+1} \\ x^r_{i+1} \end{pmatrix}$$

$$+ \begin{pmatrix} b_{i+1} \\ b^r_{i+1} \end{pmatrix} v_{i+1} + \begin{pmatrix} b_{i+1} \\ 0 \end{pmatrix} (e_i + r_i)$$

$$y_{i+1} = [c_{i+1} \quad 0] \begin{pmatrix} x_{i+1} \\ x^r_{i+1} \end{pmatrix}$$

By observability of (A^r_{i+1}, c^r_{i+1}) and by (2.15b), $r_i(\cdot)$ can be generated by

$$r_i = c^r_{i+1} \cdot \bar{x}^r_{i+1}$$

$$\dot{\bar{x}}^r_{i+1} = A^r_{i+1} \cdot \bar{x}^r_{i+1}$$

hence

$$\dot{\hat{x}}_{i+1} = \hat{A}_{i+1} \hat{x}_{i+1} + \hat{b}_{i+1} v_{i+1} + d_i$$

$$y_{i+1} = \hat{c}_{i+1} \hat{x}_{i+1}$$

with

$$\hat{x}_{i+1} = \begin{bmatrix} x_{i+1} \\ x^r_{i+1} + \bar{x}^r_{i+1} \end{bmatrix}, \quad \hat{A}_{i+1} = \begin{bmatrix} A_{i+1} & b_{i+1} \cdot c^r_{i+1} \\ 0 & A^r_{i+1} \end{bmatrix},$$

$$\hat{b}_{i+1} = \begin{bmatrix} b_{i+1} \\ b^r_{i+1} \end{bmatrix}, \quad d_i = \begin{bmatrix} b_{i+1} \\ 0 \end{bmatrix} e_i, \quad \hat{c}_{i+1} = \begin{bmatrix} c_{i+1} & 0 \end{bmatrix}$$

and $e_i(\cdot) \in M$.

Assuming observability of $(\hat{A}_{i+1}, \hat{c}_{i+1})$, $r_{i+1}(\cdot)$ can be generated by

$$r_{i+1} = \hat{c}_{i+1} \bar{x}_{i+1}, \quad \dot{\bar{x}}_{i+1} = \hat{A}_{i+1} \bar{x}_{i+1} .$$

Thus we obtain

$$(\hat{x}_{i+1}\dot{-}\bar{x}_{i+1}) = \hat{A}_{i+1}(\hat{x}_{i+1}-\bar{x}_{i+1}) + \hat{b}_{i+1}v_{i+1} + d_i$$

$$e_{i+1} = \hat{c}_{i+1}(\hat{x}_{i+1}-\bar{x}_{i+1})$$

with $d_i(\cdot) \in M$.

By Remark (2.6c), stabilization is achieved by choosing $\hat{f}_{i+1}$ and $\hat{\alpha}_{i+1}$ as before.

This gives rise to local adaptive controllers $(\hat{f}_i,\hat{\alpha}_i)$ that are constructed in the same manner as for a single system, except for the modification that instead of the polynomial $\tilde{p}_i(s)$ the polynomial $p_i(s)= \ell cm(\tilde{p}_{i-1}(s),\tilde{p}_i(s))$ is used. Thus we have proved the following

2.7 Corollary

The tracking errors $e_i(\cdot)$ in the series connection (2.13) satisfy:

$$e_i(\cdot) \in M \quad \text{for } i=1,\dots,N \tag{2.16}$$

if the systems belong to Σ_+, the reference signals $r_i(\cdot)$ belong to $R_{\tilde{p}_i(s)}$, M satisfies (Δ) and the local adaptive controllers $(\hat{f}_i,\hat{\alpha}_i)$ are of the form (2.10). If the gains $k_i(\cdot)$ are adapted by (2.5a) with p even then

$$\lim_{t\to\infty} e_i(t) = 0 \quad \text{for } i=1,\dots,N$$

□

The following simulations show asymptotic tracking for a series coupling of 8 systems with transfer functions:

$$\Sigma_1 \; : \quad g_1(s) = \frac{s+1}{s^2-2s+1}$$

$$\Sigma_2 \; : \quad g_2(s) = \frac{s^3+4s^2+5s+2}{s^4-5s^3+3s^2+4s-1}$$

$$\Sigma_3 \; : \quad g_3(s) = \frac{1}{s-1}$$

$$\Sigma_4 \; : \quad g_4(s) = \frac{s^2+2s+1}{s^3+2s^2+3s-2}$$

$$\Sigma_5 \; : \quad g_5(s) = \frac{s^4+4s^3+6s^2+4s+1}{s^5-s^4-s^3+s^2-s}$$

$$\Sigma_6 \; : \quad g_6(s) = \frac{1}{s+1}$$

$$\Sigma_7 \; : \quad g_7(s) = \frac{s+1}{s^2+2s+1}$$

$$\Sigma_8 \; : \quad g_8(s) = \frac{s^2+4s+4}{s^3+3s^2+2s-1}$$

In Fig. 1 the reference signals are

$$r_i(t) = \sin(t+\tfrac{i-1}{4}\pi), \quad i=1,\ldots,8$$

where

$$p_i(s) = s^2+1 \quad \text{and} \quad q_i(s) = (s+\pi)^2 \; .$$

In Fig. 2 for the four systems $\Sigma_1,\ldots,\Sigma_4$ the values of the constant reference signals are $(r_1,r_2,r_3,r_4) = (5,-2,2,-5)$. For both simulations the gain adaption law is: $\dot{k}(t) = e(t)^2$.

Fig. 1: Simulation of series coupling without switching functions: periodic reference signals

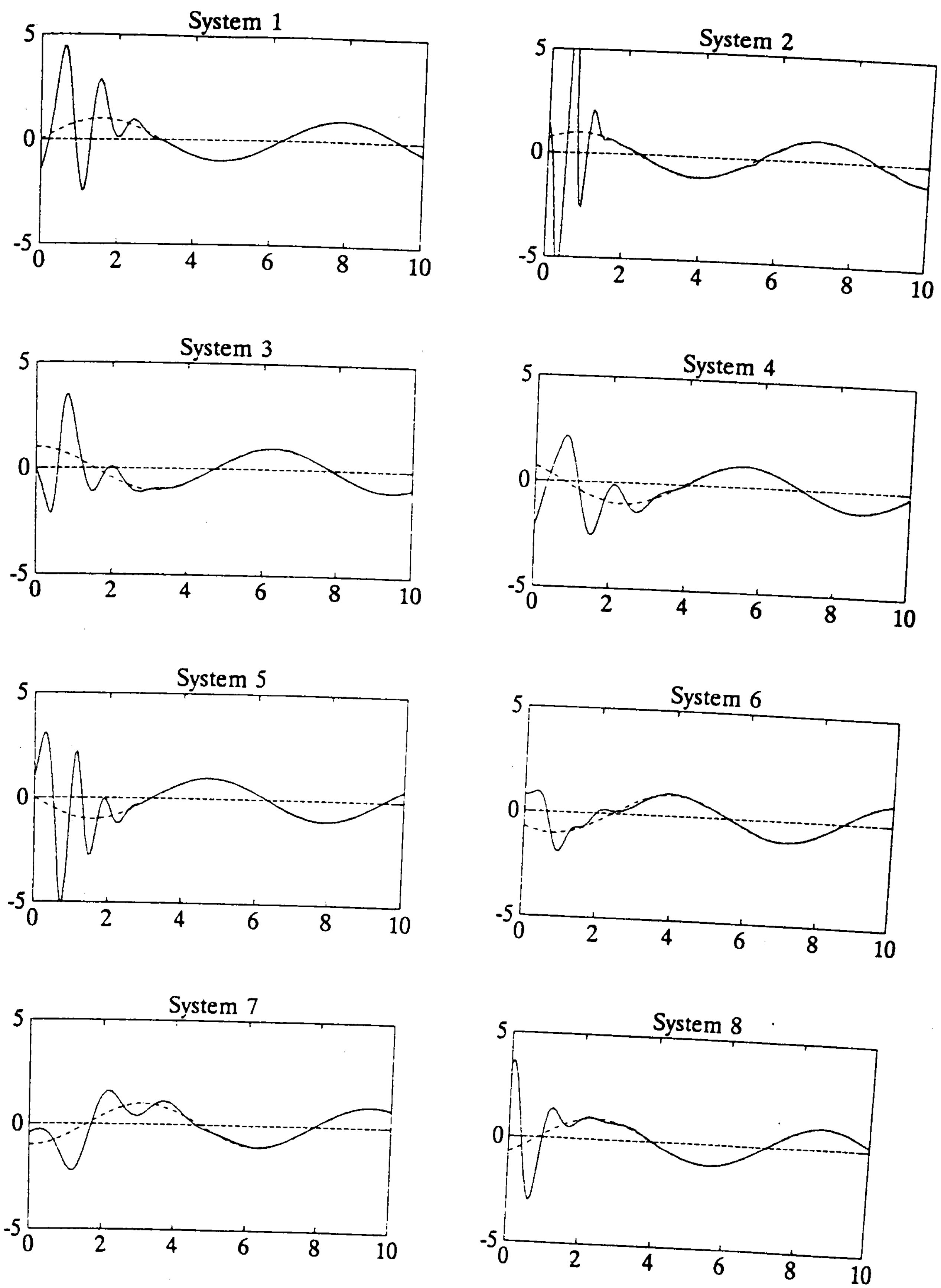
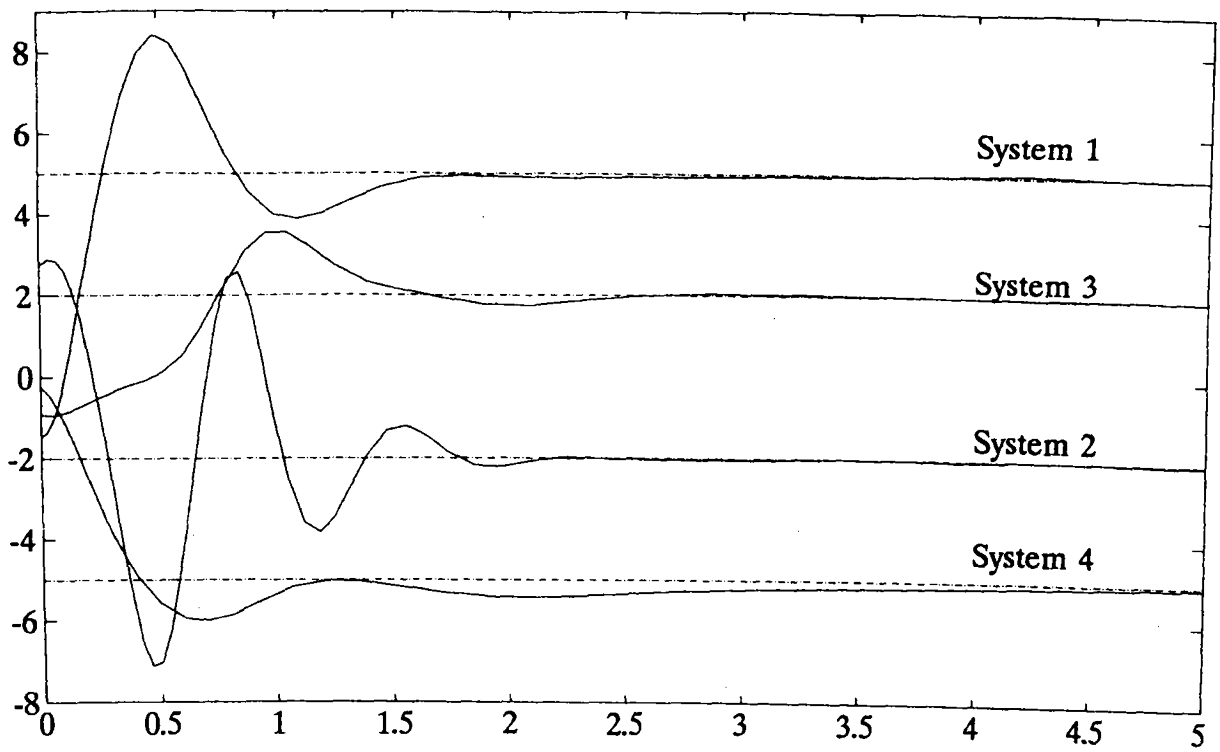


Fig. 2: Series coupling: tracking of constant reference signals



3. ROBUST ADAPTIVE TRACKING WITH SWITCHING FUNCTIONS

If UATC's are to be constructed for the class $\Sigma(n,1,1)$, i.e. if sgn cb is not known, switching concepts become necessary. The following theorem shows that the Nussbaum type switching stabilizers are robust against L_2-disturbances.

3.1 Theorem

Let $(A,b,c) \in \Sigma(n,1,1)$ be scalar minimum phase of relative degree 1 and $d(\cdot) \in L_2([0,\infty),\mathbb{R}^n)$. The solutions of the closed loop system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) + d(t) \\ y(t) &= cx(t) \\ u(t) &= N(k(t))y(t) \\ \dot{k}(t) &= y^2(t) \end{aligned} \tag{3.1}$$

where $N(\cdot)$ is a Nussbaum type switching function, i.e.

$$\sup_{\eta>0} \frac{1}{\eta}\int_0^\eta N(\sigma)d\sigma = +\infty \; , \qquad \inf_{\eta>0} \frac{1}{\eta}\int_0^\eta N(\sigma)d\sigma = -\infty \tag{3.2}$$

satisfy:

$$y(\cdot) \in L_2([0,\infty),\mathbb{R}) \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0 \tag{3.3a}$$

$$\lim_{t\to\infty} k(t) = k_\infty < \infty \tag{3.3b}$$

<u>Proof:</u> By a suitable state space coordinate transformation we can obtain the following decomposition [11]:

$$\dot{x}_1 = A_1x_1 + A_2y + d_1$$

$$\dot{y} = (\alpha+\beta N(k))y + A_3x_1 + d_2$$

where A_1 is exponentially stable and $d_1(\cdot) \in L_2$, $d_2(\cdot) \in L_2$ and

$\beta \neq 0$. Defining $\hat{x}_1$ and $\hat{d}_1$ by

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + A_2 y \ , \quad \hat{x}_1(0) = x_1(0)$$

$$\dot{\hat{d}}_1 = A_1\hat{d}_1 + d_1 \ , \quad \hat{d}_1(0) = 0 \ ,$$

we have $x_1 = \hat{x}_1+\hat{d}_1$ and $\hat{d}_1(\cdot) \in L_2$.

Multiplying the differential equation for y by $\dot{y}$ and integrating from $\tau=0$ to $\tau=t$ we get

$$\tfrac{1}{2}y^2(t) = c_1 + \int_0^t (\alpha+\beta N(k(\tau)))y^2(\tau)d\tau + \int_0^t A_3\hat{x}_1(\tau)y(\tau)d\tau$$

$$+ \int_0^t (A_3\hat{d}_1(\tau)+d_2(\tau))y(\tau)d\tau$$

$c_1 \in \mathbb{R}$ a constant.

Applying a result of ([11]) to the system

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + A_2 y$$

with output $A_3\hat{x}_1$, we obtain

$$\int_0^t |A_3\hat{x}_1(\tau)y(\tau)|d\tau \le c_2 + M\int_0^t y^2(\tau)d\tau \ , \quad c_2, M \text{ constants.}$$

Thus,

$$\tfrac{1}{2}y^2(t) \le c_2 + \int_0^t (\alpha+M+\beta N(k))y^2(\tau)d\tau + \int_0^t d_3(\tau)y(\tau)d\tau$$

with $c_3 \in \mathbb{R}$ a constant and $d_3 = A_3\hat{d}_1+d_2 \in L_2$.

Using $\int_0^t d_3(\tau)y(\tau)d\tau \le \frac{1}{2}\int_0^t d_3(\tau)^2 d\tau + \frac{1}{2}\int_0^t y(\tau)^2 d\tau$ and $d_3 \in L_2$ we obtain

$$\tfrac{1}{2}y^2(t) \le c + \int_0^t (\alpha+\tfrac{1}{2}+M+\beta N(k))y^2(\tau)d\tau$$

with $c \in \mathbb{R}$ a constant.

Proceeding now as in the proof of Thm. 2 in [11] the assumption $k(t) \to +\infty$ as $t\to\infty$ leads to a contradiction and the result follows. □

The following corollary is an immediate consequence of the above theorem and the reasoning for the construction of an UATC in the non switching case (cf. Corollary 2.5).

3.2 Corollary

Let $(A,b,c) \in \Sigma$ be minimum phase with relative degree 1, $r(\cdot) \in \mathcal{R}_{p(s)}$ and $d(\cdot) \in L_2([0,\infty),\mathbb{R}^n)$. Let further $q(s)$ be Hurwitz, $\deg q(s) = \deg p(s) = \ell$ and $(A_r,b_r,c_r,1)$ a minimal realization of $\frac{q(s)}{p(s)}$. Then the solutions $y(\cdot)$, $k(\cdot)$ of the closed loop system:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + bu(t) + d(t) \\
y(t) &= cx(t) \\
u(t) &= c_r x_r(t) + N(k(t))e(t) \\
\dot{x}_r(t) &= A_r x_r(t) + b_r N(k(t))e(t) \\
\dot{k}(t) &= e(t)^2
\end{aligned}$$

where $N(\cdot)$ is a Nussbaum type switching function, satisfy:

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (y(t)-r(t)) = 0$$

$$k_\infty = \lim_{t\to\infty} k(t) < \infty \quad \text{exists.}$$

□

Now consider the series connection (2.13), where the subsystems (A_i,b_i,c_i) are allowed to belong to the larger class $\Sigma(n_i,1,1)$ instead of $\Sigma_+(n_i,1,1)$. Change the local controllers (2.10), (2.14) by replacing (2.10a) by the control law

$$u(t) = c_r x_r(t) + N(k(t))e(t) \tag{2.10'a}$$

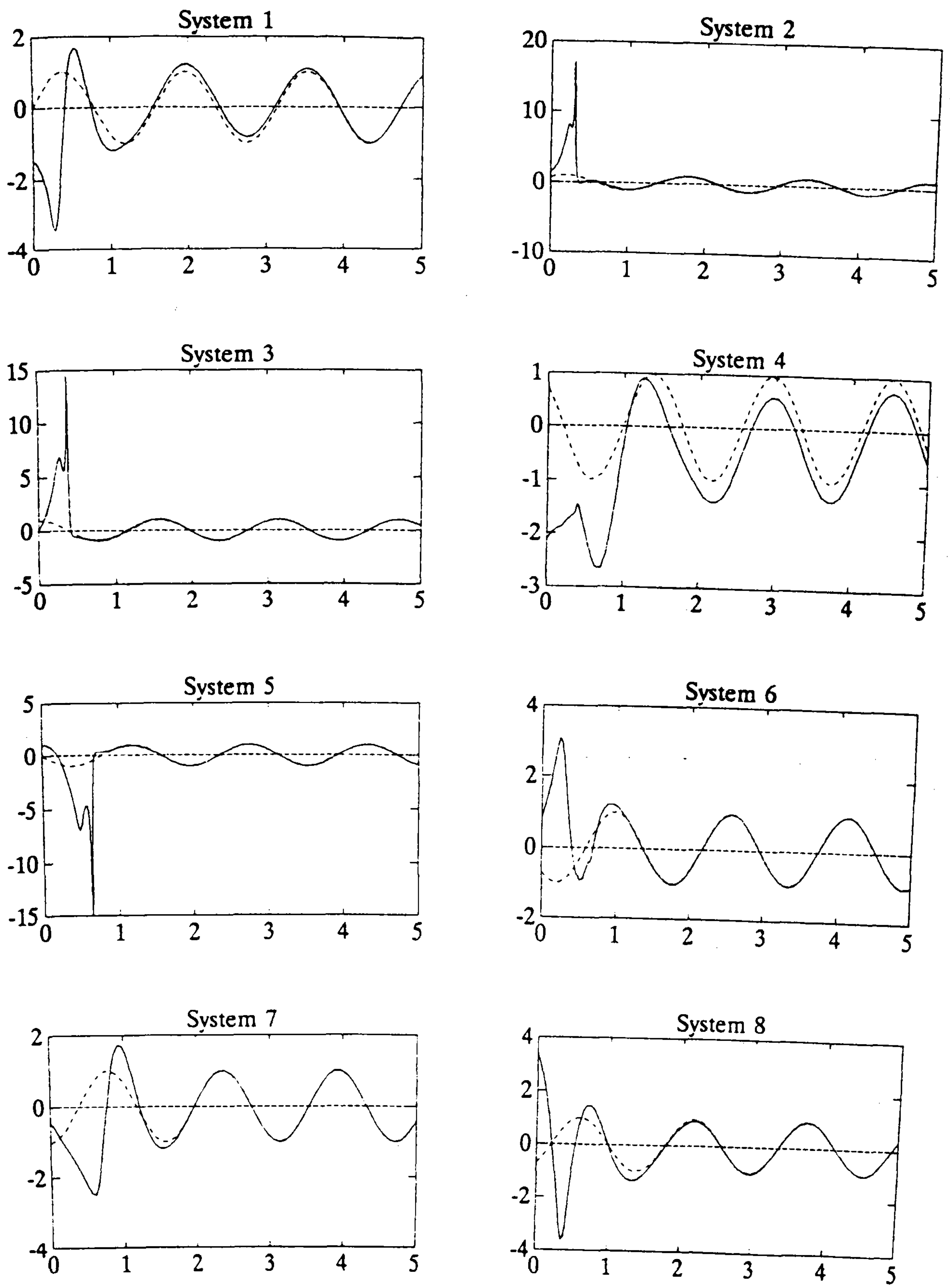
for the i-th subsystems. Using Theorem (3.1), the same arguments as for Corollary (2.7) show that the local tracking errors $e_i(t)$ and gains $k_i(t)$ for the closed loop system satisfy

$$\lim_{t\to\infty} e_i(t)$$

$$\lim_{t\to\infty} k_i(t) = k_{i,\infty} \in \mathbb{R} \quad \text{exists.}$$

The following simulations clearly demonstrate this behavior. The 8 systems are the same as in Fig. 1. The reference signals are $r_i(t) = \sin(4t+\frac{i-1}{4}\pi)$ and the switching function is $N(k) = k^2 \cos k$.

Fig. 3: Series coupling: tracking of periodic signals by switching controllers



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